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STOCHASTIC TACHYON FLUCTUATIONS, MARGINAL DEFORMATIONS AND SHOCK WAVES IN STRING THEORY

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ABSTRACT

Starting with exact solutions to string theory on curved spacetimes we obtain deformations that represent gravitational shock waves. These may exist in the presence or absence of sources. Sources are effectively induced by a tachyon field that randomly fluctuates around a zero condensate value. It is shown that at the level of the underlying conformal field theory (CFT) these deformations are marginal and moreover all α'-corrections are taken into account. Explicit results are given when the original undeformed 4-dimensional backgrounds correspond to tensor products of combinations of 2-dimensional CFT's, for instance $SL(2,\mathbb{R})/\mathbb{R} \otimes SU(2)/U(1)$.

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1. Introduction

Gravitational shock waves in general relativity have been considered in depth in the past as well as more recently, with the prototype example being the shock wave due to a massless particle moving in a flat Minkowski background [1]. The generalization to the case where the particle moves along a null hypersurface of a more general class of vacuum solutions to Einstein's equations was found in [2] and for the cases where there are non-trivial matter fields and a cosmological constant in [3]. Explicit results were given when the curved background geometry is the Schwarzschild black hole in [2], and for the cases of the Reissner-Nordström charged black hole, the De-Sitter space, and the Schwarzschild-de-Sitter black hole in [3]. Other interesting solutions representing the gravitational field of massless particles with extra quantum numbers (charge, spin), cosmic strings, or monopoles in a flat Minkowski background [4], or in De-Sitter space [5], have been obtained by infinitely boosting [1,2] known solutions representing curved spacetimes. For the cases where instead of a massless particle there is a distribution of massless matter, such as spherical and planar shells, see [6,7].

The main motivation for dealing with gravitational shock waves is that, as was argued in [8], gravitational interactions dominate any other type of interaction at Planckian energies (see [9,10,11,12]) and that in an S-matrix approach to black hole physics [8], one needs to take into account the interactions between Hawking emitted and infalling particles as well as their effect on the original black hole geometry. Thus, having the exact solutions to Einstein's equations (and for that matter to any other theory of gravity) of a background geometry coupled to a distribution of massless matter moving along a null hypersurface is equivalent to fully taking into account all classical backreaction-type effects.

The purpose of this paper is to analyze gravitational shock waves in the context of string theory. This was partially done in [3], but from a general relativity point of view. However, as we shall see, the origin of such solutions in string theory is different from that in general relativity. Moreover, new features will be found and a direct connection

with the underlying conformal field theory (CFT) will be made. The paper is organized as follows: In section 2 we develop the necessary formalism and obtain the general condition for being able to introduce a shock wave in a quite general class of solutions to string theory with two different methods. One is the general relativity inspired traditional method [2], where one essentially solves the β-function equations assuming that they are satisfied by the background geometry fields. The second method, which is new and uses CFT techniques, reveals that the shock waves correspond to marginal perturbations of the CFT corresponding to the original background. It yields the same results as the more traditional method in a straightforward way requiring however much less effort. In addition, as we shall see, it is applicable to more general situations. We also show that random fluctuations of the tachyon field around its zero average value effectively produce source terms for gravitational shock waves, which nevertheless may exist even in the absence of sources. In section 3 we apply the general formalism to several cases where the background fields correspond to tensor products of various combinations of 2-dimensional exact CFT's. We end the paper with concluding remarks and a discussion in section 4. In order to facilitate the computations of section 2 we have written appendix A containing components of various useful tensors and appendix C containing elements of stochastic calculus. In Appendix B we use the CFT method to find shock waves on more general backgrounds than the ones considered in section 2.

2. General formalism and results

Consider the string background in d spacetime dimensions that comprises a metric, an antisymmetric tensor, and a dilaton field given by

$$\begin{aligned} ds^2 &= 2\ A(u,v)\ dudv\ +\ g(u,v)\ h_{ij}(x)\ dx^i dx^j \\ B &= 2B_{ui}(u,v,x)\ du\wedge dx^i + B_{ij}(u,v,x)\ dx^i\wedge dx^j \\ \Phi &= \Phi(u,v,x)\ , \end{aligned} \tag{2.1}$$

with $(i,j=1,2,\ldots,d-2)$. Let us suppose that a 'disturbance' is introduced (whose origin and nature will be examined later in this section) with the net effect that the spacetime is described by (2.1) only for $u<0$, whereas for $u>0$ we should replace in (2.1) $v\to v+f(x)$ and $dv\to dv+f_{,i}dx^i$. Thus the two spacetimes for $u<0$ and $u>0$ are glued together along the null hypersurface $u=0$ [2]. A compact way to represent the spacetime fields is by using the Heaviside step function $\vartheta=\vartheta(u)$

$$
\begin{aligned}
&ds^2 = 2\ A(u,v+\vartheta f)\ du(dv+\vartheta f_{,i}dx^i)\ +\ g(u,v+\vartheta f)\ h_{ij}(x)\ dx^i dx^j\\
&B = 2B_{ui}(u,v+\vartheta f,x)\ du\wedge dx^i + B_{ij}(u,v+\vartheta f,x)\ dx^i\wedge dx^j\\
&\Phi = \Phi(u,v+\vartheta f,x)\ .
\end{aligned}
\tag{2.2}
$$

The coordinate change

$$
u\to u\ ,\quad v\to v-f(x)\vartheta(u)\ ,\quad x^i\to x^i\ ,
\tag{2.3}
$$

gives a form where various tensors are easier to compute

$$
\begin{aligned}
&ds^2 = 2\ A(u,v)\ dudv + g(u,v)\ h_{ij}(x)\ dx^i dx^j + F(u,v,x)\ du^2\\
&B = 2B_{ui}(u,v,x)\ du\wedge dx^i + B_{ij}(u,v,x)\ dx^i\wedge dx^j\\
&\Phi = \Phi(u,v,x)\ ,
\end{aligned}
\tag{2.4}
$$

where $F(u,v,x)\equiv -2A(u,v)f(x)\delta(u)$, and $\delta(u)=\frac{d\vartheta(u)}{du}$ is a δ-function. In order to determine the shift function $f(x)$ we require that the β-function equations that govern the dynamics of the lowest modes of the string are satisfied[1]

$$
\begin{aligned}
&R_{\mu\nu} - D_\mu D_\nu\Phi - \frac{1}{4}H_{\mu\rho\lambda}H_\nu{}^{\rho\lambda} = \partial_\mu T\partial_\nu T\\
&D_\lambda(e^\Phi H^\lambda{}_{\mu\nu}) = 0\\
&C = d + \frac{3}{2}\alpha'\big(D^2\Phi + (D\Phi)^2 - \frac{1}{6}H^2_{\mu\nu\lambda} - (DT)^2 - 2V(T)\big)\\
&D^2T + D_\mu\Phi D^\mu T = V'(T)\ ,
\end{aligned}
\tag{2.5}
$$

[1] For zero tachyon field ($T=0$) these are the one loop β-function equations corresponding to the metric, antisymmetric tensor, and dilaton fields (see, for instance, [13,14]). The tachyonic contributions [15] are non-perturbative in the loop expansion.

where the tachyon potential is $V(T) \sim T^2$ and C denotes the central charge. By assumption, in the bulk of the space ($u \neq 0$) and with zero tachyon field, i.e., $T = 0$, these equations are automatically satisfied by the background fields (2.4) or equivalently (2.1). However, as one might expect, there are extra contributions from the boundary at $u = 0$ (in fact multiplied by a $\delta(u)$-function). Using the results of appendix A we find that consistency requires the conditions

$$A_{,v} = g_{,v} = \Phi_{,v} = 0 \quad \text{at } u = 0 \ . \tag{2.6}$$

In addition the shift function $f(x)$ is obtained by solving the linear differential equation[2]

$$\begin{aligned} &\triangle f(x) - c(x) f(x) = 2\pi b \ \delta^{(d-2)}(x - x') \\ &c(x) \equiv \frac{1}{A}\Big(\frac{d-2}{2} g_{,uv} + g\Phi_{,uv}\Big) \ , \quad b = k\frac{g}{A} \ , \end{aligned} \tag{2.8}$$

where all functions are computed at $u = 0$ and the Laplacian is defined as

$$\triangle = \frac{1}{e^{\Phi}\sqrt{h}} \partial_i e^{\Phi} \sqrt{h} \ h^{ij} \partial_j \ . \tag{2.9}$$

The conditions (2.6)(2.8) were derived by examining the metric β-function. The rest of the equations give no additional information.[3] We have also included a specific source term (with strength proportional to a constant k) containing a δ-function defined in the

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[2] In order to cast it into that form we have used the fact that

$$\frac{A_{,uv}}{A} + \frac{d-2}{2}\frac{g_{,uv}}{g} + \Phi_{,uv} - \frac{1}{2gA} H_{uvi} H_{uvj} h^{ij} = 0 \quad \text{at } u = 0 \ . \tag{2.7}$$

This is nothing but the (u, v)-component of the metric β-function computed at $u = 0$ and simplified by using (2.6) and the fact that $H_{vij} = 0$ at $u = 0$ (this follows from the (v, v) and (v, i)-component of the metric β-function).

[3] Had we included a B_{vi} component in the antisymmetric tensor we should have required that, in addition to (2.6)(2.8), $B_{vi} = O(u^n)$ and $B_{vi,v} = O(u^m)$, with $n > \frac{1}{2}$ and $m > 1$, near $u = 0$. If $B_{vi,v} = O(u)$ an additional non-linear term $O(f^3)$ seems to appear in (2.8). Since this could be a new feature it might be interesting to explore it further.

transverse x^i-space and normalized with respect to the 'string measure' $e^{\Phi}\sqrt{h}$ computed at $u=0$, namely,

$$\int_{M_\perp} d^{d-2}x \; e^{\Phi}\sqrt{h}\; \delta^{(d-2)}(x) = 1 \;, \quad \text{at } u=0 \;. \tag{2.10}$$

The conditions (2.6) and (2.8) are the string theory analogue of the similar conditions found in the context of Einstein's general relativity in d-dimensions [3] and they reduce to them for a constant dilaton field.

So far we have given no explanation at all about the origin of the source term present on the right hand side of the equation in (2.8). Strictly speaking for a zero tachyon field it should be zero. In fact in certain cases (but not in general) that differential equation with the zero right hand side has a solution. It can be shown (see also [3]) that in such cases the term

$$-2\int d^2z \; \delta(u) f(x) A(u,v)\; \partial u \bar{\partial} u \;, \tag{2.11}$$

corresponds to a marginal perturbation of the original 2-dimensional σ-model action for the background (2.1) in the sense that it solves the corresponding conditions as they were found, to leading order in α', in [16] (in this paper it was assumed that the antisymmetric tensor was identically zero, but presumably a generalization to the non-zero case exists). In fact it can be shown that these conditions reduce, in our case, to just (2.6)(2.8). Before we explain the origin of the δ-function source term on the right hand side of (2.8), let us rederive (2.6)(2.8) using standard CFT techniques. We will show that (2.6)(2.8) are the necessary and sufficient conditions for (2.11) to be a marginal perturbation (in fact we will argue that is is exactly marginal) and that these conditions hold beyond the one loop approximation, i.e., are in fact exact to all orders in α'. This method is considerably faster and could be easily adopted to other similar situations (see appendix B). The first step is to show that ∂u has conformal dimension $(1,0)$ with respect to the energy momentum tensor corresponding to the background (2.1) in the limit $u \to 0$ (remember the $\delta(u)$-function). The holomorphic component of the energy momentum tensor is

$$T = \frac{1}{\alpha'}\big(: A\partial u \partial v : + : g h_{ij}\partial x^i \partial x^j : \big) + : \partial^2 \Phi : \;, \tag{2.12}$$

where a proper regularization prescription is implied. In general finding the operator product expansions (OPE's) for the fields u, v, x^i is very difficult due to non-linearities. However, close to $u = 0$ we can infer that

$$u(z,\bar{z})v(w,\bar{w}) = \frac{1}{A}\ln|z-w|^2 + \dots\ , \tag{2.13}$$

where the ellipsis denotes terms that vanish as $z \to w$ (and $\bar{z} \to \bar{w}$). Therefore ∂u at $u = 0$ will be a dimension $(1,0)$ operator with respect to (2.12) and its antiholomorphic partner, provided that all possible anomalies arising from contractions with the fields u, v in $A(u,v)$, $g(u,v)$ and $\Phi(u,v)$ vanish. It is easily seen that conditions (2.6) guarantee exactly that. Rephrasing, conditions (2.6) guarantee that close to $u = 0$ the CFT for the longitudinal part is effectively that of two free bosons. Analogously $\bar{\partial} u$ at $u = 0$ has dimension $(0,1)$ and thus the operator $\partial u \bar{\partial} u$ has dimension $(1,1)$ at $u = 0$. Having established that, we need to discover the condition $A(u,v)f(x)\delta(u)$ has to satisfy in order to really be a function, i.e., have dimension $(0,0)$. Then the term (2.11) will correspond to a marginal perturbation (but not in principle exactly marginal). On general grounds, the dimension D of $A(u,v)f(x)\delta(u)$ is determined from the eigenvalue, Klein-Gordon type, equation

$$-\frac{1}{e^{\Phi}\sqrt{-G}}\partial_\mu e^{\Phi}\sqrt{-G}G^{\mu\nu}\partial_\nu\ A(u,v)f(x)\delta(u) = D\ A(u,v)f(x)\delta(u)\ , \tag{2.14}$$

where $\sqrt{-G} = Ag^{\frac{d-2}{2}}\sqrt{h}$. Demanding that $D = 0$ and after simplifying using (2.6), the above equation reduces exactly to (2.8) with a zero right hand side.

So far we have set the tachyon field to zero. However, we can slightly relax this condition by demanding that only its average value is zero but otherwise it can randomly fluctuate. The origin of such fluctuations can be either statistical, due to our inability to determine its precise value, or of more fundamental nature, as remnants of stringy phenomena at high energies (ambiguity in the usual spacetime description at Planck scale, etc.) that effectively make, at low energies, the tachyon to appear fluctuating, or a combination of both. We will show that these fluctuations induce (upon taking the average of

the tachyon energy momentum tensor) the non-zero source term on the right hand side of (2.8). Specifically let us consider a tachyon T that factorizes as

$$T(u,x) = \tau(u)\ h(u,x)\ , \tag{2.15}$$

where $\tau(u)$ is given by the stochastic integral in the Ito calculus,

$$\tau(u) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} dw(t)\ e^{itu}\ , \tag{2.16}$$

and $h(u,x)$ is any deterministic function that behaves as

$$h(u,x) = \sqrt{\rho(x)}\ u + O(u^2)\ , \tag{2.17}$$

where $\rho(x)$ is a density-like function. Elements of stochastic calculus are given in appendix C. Using them we compute the expectation values

$$\begin{aligned}
&\langle \tau^2(u) \rangle = 2\delta(u) \\
&\langle \tau(u)\tau'(u) \rangle = \delta'(u) \\
&\langle \tau'^2(u) \rangle = \frac{1}{2}\delta''(u)\ ,
\end{aligned} \tag{2.18}$$

where the prime denotes differentiation with respect to u. From these and the leading order behavior of $h(u,x)$ near $u=0$ we obtain expectation values involving directly the tachyon field

$$\begin{aligned}
&\langle T \rangle = 0 \\
&\langle \partial_i T \partial_j T \rangle = 2\ \partial_i \sqrt{\rho(x)} \partial_j \sqrt{\rho(x)}\ u^2 \delta(u) = 0 \\
&\langle \partial_u T \partial_i T \rangle = \frac{1}{2} \partial_i \rho(x) (u^2 \delta'(u) + 2u\delta(u)) = 0 \\
&\langle (\partial_u T)^2 \rangle = \rho(x) \big(2\delta(u) + 2u\delta'(u) + \frac{1}{2} u^2 \delta''(u) \big) = \rho(x)\delta(u)\ .
\end{aligned} \tag{2.19}$$

It is understood that (2.19) hold in a distribution sense with respect to the variable u and therefore integration over a smooth function of u is implied.

The upshot of this analysis is that by taking the expectation value of the β-function equations[4] and by using (2.19) we obtain the same equations we would have obtained had we set the tachyon field to zero *except* from a source term on the right hand side of the $\mu = \nu = u$ component of the metric β-function. Namely

$$R_{uu} - D_u D_u \Phi - \frac{1}{4}(H^2)_{uu} = \rho(x)\delta(u) \ , \tag{2.20}$$

thus proving that random fluctuations around a zero tachyon background induce source terms for gravitational shock waves in string theory.[5] Since we are interested in the Green's functions we chose $\rho(x) = k \ \delta^{(d-2)}(x - x')$. The result for any other distribution $\rho(x)$ is given by the integral $\int [dx'] f(x, x')\rho(x')$, where the measure $[dx']$ is defined according to (2.10). Let us also mention that the central charge coincides with the value obtained by simply having a vanishing tachyon. Not only do the tachyon dependent terms vanish upon taking the average but also the derivatives of them with respect to all fields (including u) as well. Also it is obvious that in this case the term (2.11) does not represent a marginal perturbation by itself. However, its conformal anomaly balances that of the stochastically fluctuating tachyon. This is precisely the meaning of (2.8).

In the CFT approach to deriving (2.6)(2.8) the backgrounds (2.1)(2.4) are supposed to satisfy the β-function equations to all orders in perturbation theory in powers of α' in the standard 'conformal scheme' (see [17]), where also the tachyon equation takes the simple form given by the last equation in (2.5). Thus, we conclude that (2.6)(2.8) are indeed valid to all orders in conformal perturbation theory in the standard 'conformal scheme'.

The final comment is on whether or not (2.11) represents an exactly marginal perturbation. This would be the case if we can argue that higher order terms in f do not spoil

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[4] Notice that without taking the average the β-functions are not satisfied due to the tachyon stochastic fluctuations. However, our philosophy is that they need only be satisfied in the 'average'.

[5] Tachyon fluctuations are not the only possible such source. For instance, if $A(u, v)$, $g(u, v)$ are constant functions and $\Phi(u, v, x) = \Phi(x) + ku\vartheta(u)$ we obtain a source term with uniform distribution $\rho(x) = k = \text{const.}$, because in this case $D_u D_u \Phi = k\delta(u)$.

conformal invariance. In fact such non-linearity in f terms does appear when we consider the β-function equations (see (A.2),(A.3)). However, in that case one can show that these terms vanish in a distribution sense [3]. In the present case we can argue that anomaly terms proportional to powers of f in the Virasoro algebra generated by the energy momentum tensor corresponding to the background (2.4) also vanish, as follows. A possible such term contains the factor

$$\prod_i \frac{\partial^{n_i+n'_i} A}{\partial v^{n_i} \partial u^{n'_i}} \; \frac{\partial^{m_i+m'_i} g}{\partial v^{m_i} \partial u^{m'_i}} \; \frac{\partial^{l_i+l'_i} \Phi}{\partial v^{l_i} \partial u^{l'_i}} \; f^n \delta^n \ , \tag{2.21}$$

where the relation between the various integers

$$\sum_i (n_i + m_i + l_i) = 2n + \sum_i (n'_i + m'_i + l'_i) \geq 2n \tag{2.22}$$

should also hold. This follows from the facts that near $u = 0$ there is an invariance of the theory described by (2.1) under u, v interchange (cf.(A.1)-(A.4)) and that the energy momentum tensor must be invariant under this symmetry. Since for regular functions around $u = 0$ each derivative with respect to v contributes a power of u, it can easily be seen that (2.21) vanishes as a distribution thanks to the inequality in (2.22). Let us once more emphasize that the term (2.11) corresponding to an exactly marginal perturbation is a consequence of (2.6)(2.8) and the presence of the $\delta(u)$-function, and that this is not generally true for marginal perturbations with abelian chiral currents [18].

3. Applications

In all of our applications we start with the direct product of two 2-dimensional CFT's with a metric and dilaton of the form (the antisymmetric tensor is zero)

$$\begin{aligned} ds^2 &= 2A(u,v)\ dudv + h_{ij}(x)\ dx^i dx^j \\ \Phi(u,v,x) &= \phi_{\|}(u,v) + \phi_{\perp}(x^1,x^2)\ , \end{aligned} \tag{3.1}$$

namely, the longitudinal and the transverse parts are decoupled. The longitudinal CFT provides the 'time' coordinate for the metric of our model and we will take it to be either

the one corresponding to the coset $SL(2,\mathbb{R})_{-k}/\mathbb{R}$ [19,20], or that corresponding to the flat 2-dimensional Minkowski space.[6] As the transverse part we will take either the compact coset $SU(2)_k/U(1)$ [23] or flat 2-dimensional space (with possibly a linear dilaton) or the dual to the 2-dimensional flat space. The coupling between the two CFT's is only due to the term (2.11) corresponding to the shift function $f(x)$ which satisfies the differential equation (2.8), with constant c. The solution to this differential equation can be expressed as an infinite sum over eigenfunctions of the Laplacian (2.9). The result is easily found to be (we ignore possible solutions of the homogeneous equation)

$$f(x;x') = -2\pi b \sum_N \frac{\Psi_N(x)\Phi^*_N(x')}{E_N + c} \ , \tag{3.2}$$

where we denoted by N all possible quantum numbers arising from the eigenvalue equation

$$\triangle \Psi_N(x) = -E_N \Psi_N(x) \ . \tag{3.3}$$

Notice that since we are dealing with compact manifolds corresponding to the transverse metric the Laplacian (2.9) is a negative definite operator. For this reason the minus sign on the right hand side of (3.3) implies that $E_N \geq 0$. The eigenfunctions satisfy the orthonormalization condition

$$\int_\Sigma d^2x e^{\phi_\perp} \sqrt{h} \Psi_N(x) \Phi^*_M(x) = \delta_{N,M} \ , \tag{3.4}$$

and the completeness relation

$$\sum_N \Phi_N(x)\Phi^*_N(x') = \delta^{(2)}(x - x') \ . \tag{3.5}$$

Notice that, in accordance with (2.9) and (2.10) the 'string measure' $e^{\phi_\perp}\sqrt{h}$ has been used and not just $\sqrt{h}$. Also that, in the sourceless case, (3.3) is exactly of the form (2.8).

[6] It can be shown that if we take as the longitudinal CFT the one corresponding to the dual of the 2-dimensional Minkowski space (corresponding to the coset $E^c_2/(\mathbb{R} \otimes U(1))$ [21,22]) the conditions (2.6) are not satisfied.

Therefore, if $c < 0$ and moreover coincides with one of the eigenvalues, i.e., $c = -E_N$ for some N, the corresponding eigenfunction Ψ_N gives the solution for the shift function f. In the case with source, solution (3.2) is not valid if c coincides with any of the eigenvalues E_N. Then the solution will be given in terms of the 'partner' of the Ψ_N in (3.3) which has the appropriate singular short distance behaviour that produces a δ-function (For specific examples see [3]. An important one is gravitational shock waves with sources in 4-dimensional De-Sitter space). These cases will not be considered in this paper.

Let us also mention that for notation, conventions, and various results involving special functions we will use [24].

3.1. $SL(2,\mathbb{R})_{-k}/\mathbb{R}\otimes\mathbb{R}\otimes\mathbb{R}'$

For this model the metric and dilaton are

$$\begin{aligned}
&ds^2 = \frac{2\epsilon}{uv-1}\, dudv + a^2(dx_1^2 + dx_2^2) \\
&\Phi = \phi_{\|}(u,v) + \phi_{\perp}(x_1)\ , \qquad \phi_{\|} = \ln(1-uv),\ \phi_{\perp} = 2\alpha_0 x_1 \\
&-\infty < u, v, x^1, x^2 < \infty\ .
\end{aligned} \tag{3.6}$$

where a is a constant and $\epsilon = \mathrm{sign}(k)$, with $(-k)$ being the central extension of the $SL(2,\mathbb{R})_{-k}$ current algebra.[7] For $\epsilon = 1$ the causal structure of the spacetime is that of a black hole [20] with a singularity at future times $t = u+v$, whereas for $\epsilon = -1$ it has the cosmological interpretation of an expanding Universe with no singularity at future times $t = u - v$ (see [27]). We have also allowed for the possibility of a linear dilaton in the transverse part with strength proportional to the constant α_0. In this case the eigenvalue equation is

$$\partial_{x^i}\partial_{x^i}\Psi + 2\alpha_0\partial_{x^1}\Psi = -E\Psi\ , \tag{3.7}$$

[7] Even though (2.6)(2.8) are exact expressions, for simplicity of the presentation we have chosen to work in the small α' limit corresponding to a high level ($k >> 1$) current algebra. The same remark holds for the other examples we consider in this section. The exact, in α', expressions for f, in the standard 'conformal scheme' (see [17]), can be found using the results of [25,26].

which after substituting $\Psi \to e^{-\alpha_0 x^1}\Psi$ becomes the eigenvalue equation for the standard Laplace operator in Euclidean flat space with $E \to E - a_0^2$. Therefore the eigenfunctions and eigenvalues (in a definite angular momentum sector when $\alpha_0 = 0$) are

$$\Psi_{k,m}(\rho,\phi) = \frac{1}{\sqrt{2\pi}}\; e^{-\alpha_0\rho\cos\phi}\; e^{im\phi}\; J_m(k\rho)\ , \quad E_k = k^2 + \alpha_0^2\ . \tag{3.8}$$

If we denote by $c \equiv \alpha_0^2 + \epsilon a^2$ then the solution for the shift function is

$$f(\rho,\phi;\rho',\phi') = -b\; e^{-\alpha_0(\rho\cos\phi+\rho'\cos\phi')} \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} dk \frac{k}{k^2+c} e^{im(\phi-\phi')} J_m(k\rho)J_m(k\rho')\ , \tag{3.9}$$

which after using a resummation theorem for Bessel functions and computing an integral becomes

$$f(\rho,\phi;\alpha',\phi') = b\; e^{-\alpha_0(\rho\cos\phi+\rho'\cos\phi')} \begin{cases} -K_0(\sqrt{c}R)\ , & \text{if} \quad c>0 \\ \ln(R)\ , & \text{if} \quad c=0 \\ \frac{\pi}{2}N_0(\sqrt{|c|}R)\ , & \text{if} \quad c<0\ , \end{cases} \tag{3.10}$$

where $R = \sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}$. Another way to obtain the same result without resorting to the general prescription (3.2) is by noticing that after substituting $f \to e^{-\alpha_0 x^1} f$ the equation for f is either the Bessel equation (if $c<0$) or the modified Bessel one (if $c>0$) and that the special functions in (3.10) are the only solutions with the appropriate logarithmic behavior that produces the δ-function. Let us mention that the case of flat space in the longitudinal part corresponds to letting $\epsilon = 0$ in (3.10) (but not in (3.6)) be zero. This is in fact the analogue of the result of [1] for string theory. Notice however that the presence of the linear dilaton in the transverse part modifies the solution which now depends explicitly on ϕ and also it vanishes exponentially for large R's instead of growing logarithmically.

In the sourceless case for $c<0$ ($c>0$) a basis of solutions of (2.8) for f is given by (3.8) with the replacement $J_m(k\rho) \to J_m(\sqrt{|c|}\rho)$ ($I_m(\sqrt{c}\rho)$). For the case $c=0$ ($a=\alpha_0$) a solution is $f(x^1,x^2) = e^{-\alpha_0 x^1}\big((x^1)^2-(x^2)^2\big)$.

3.2. $SL(2,\mathrm{I\!R})_{-k}/\mathrm{I\!R} \otimes$ *(dual to 2d flat space)*

In this case the metric and dilaton are

$$
\begin{aligned}
& ds^2 = \frac{2\epsilon}{uv-1}\ dudv + a^2(d\rho^2 + \frac{4}{\rho^2}\ d\phi^2) \\
& \Phi = \phi_{\|}(u,v) + \phi_{\perp}(\rho)\ , \qquad \phi_{\|} = \ln(1-uv),\ \phi_{\perp} = \ln(\rho^2) \qquad (3.11) \\
& -\infty < u,v < \infty\ , \quad 0 < \rho < \infty\ , \quad \phi \in [0,2\pi]\ ,
\end{aligned}
$$

where a is a constant, $\epsilon = \mathrm{sign}(k)$ and the physical interpration of (3.11) is similar with that in the previous example. The background in (3.11) can be obtained if we write the transverse part of the background (3.6) (with $\alpha_0 = 0$) in terms of polar coordinates and then perform a duality transformation with respect to ϕ. Notice that there is no linear dilaton term (as in (3.6)) for the transverse part since that would not be consistent with conformal invariance. The eigenvalue equation to be solved is

$$
\frac{1}{\rho}\partial_\rho \rho \partial_\rho \Psi + \frac{1}{4}\rho^2 \partial_\phi^2 \Psi = -E\Psi\ . \qquad (3.12)
$$

Changing variables as $\rho^2 = \xi$ and substituting

$$
\Psi(\rho,\phi) = e^{im\phi} e^{-\frac{1}{4}|m|\xi} T(\xi)\ , \quad m \in Z \qquad (3.13)
$$

we see that $F(\xi)$ satisfies the Laguerre equation

$$
\xi F'' + (1 - \frac{1}{2}|m|\xi)F' + \frac{1}{4}(E - |m|)F = 0\ . \qquad (3.14)
$$

This has as solution Laguerre polynomials provided that $E = |m|(2n+1)$. Therefore the appropriately normalized eigenfunctions and the corresponding eigenvalues are

$$
\begin{aligned}
& \Psi_{n,m}(\rho,\phi) = \sqrt{\frac{|m|}{2\pi}}\ e^{im\phi}\ e^{-\frac{1}{4}|m|\rho^2}\ L_n(\frac{1}{2}|m|\rho^2) \\
& E_{n,m} = |m|(2n+1)\ , \quad n = 0,1,\ldots,\ m \in Z - \{0\}\ .
\end{aligned} \qquad (3.15)
$$

Notice that we have excluded the value $m = 0$ since the corresponding eigenfunctions and eigenvalues become zero. In addition to the discrete part of the spectrum there is also a

continuous one exactly when $m = 0$. One way to see that is to cast (3.12) in the form of a Schrödinger equation and read off the corresponding effective potential, which turns out to be $V(\rho) = \frac{1}{8}(m^2\rho^2 - \rho^{-2})$. On general grounds for $m \neq 0$ there are only the bound states with the discrete eigenvalues (3.15) whereas for $m = 0$ the spectrum is continuous with[8]

$$\Psi_k(\rho) = \frac{1}{\sqrt{2\pi}}\, J_0(k\rho) \ , \qquad E_k = k^2 \ . \tag{3.17}$$

Let us point out that had we missed (3.17) we would not be able to write down the completeness relation (3.5). The solution for the shift function after we compute an integral is

$$f(\rho,\phi;\rho',\phi') = f_0(\rho;\rho') - 2b \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{m\cos m(\phi-\phi')}{m(2n+1)+c} e^{-\frac{1}{4}m(\rho^2+\rho'^2)} L_n(\frac{1}{2}m\rho^2) L_n(\frac{1}{2}m\rho'^2) \ , \tag{3.18}$$

where $c = \epsilon a^2$ and

$$f_0(\rho;\rho') = b \begin{cases} -I_0(\sqrt{c}\rho_<)K_0(\sqrt{c}\rho_>) \ , & \text{if} \quad c > 0 \\ \frac{\pi}{2} J_0(\sqrt{|c|}\rho_<)N_0(\sqrt{|c|}\rho_>) \ , & \text{if} \quad c < 0 \ , \end{cases} \tag{3.19}$$

is the ϕ-independent part of the solution. In the case of $c = 0$ (corresponding to taking flat Minkowski space in the longitudinal part) the solution is found to be

$$f(\rho,\phi;\rho',\phi') = b\,\big(\ln R - \sum_{m=1}^{\infty} \cos m(\phi-\phi') I_0(\frac{1}{4}m\rho_<^2) K_0(\frac{1}{4}m\rho_>^2)\big) \ , \tag{3.20}$$

where, as usual, $\rho_<$ ($\rho_>$) denotes the smaler (larger) of ρ, ρ'.

As an important remark let us mention that although certain backgrounds might be related via a duality transformation, the presence of the term (2.11) destroys (in general)

---

[8] In addition to directly solving (3.12) with $\partial_\phi\Psi = 0$, there is another way to obtain it from (3.15). Namely, by letting $m \to 0$, $n = \frac{1}{2}k^2/m \to \infty$, $x = \frac{1}{4}k^2\rho^2$, and using

$$\lim_{n\to\infty} L_n(\frac{x}{n}) = J_0(2\sqrt{x}) \ . \tag{3.16}$$

this relationship. An example is that although (3.6) (with $\alpha_0 = 0$) and (3.11) are duality related, after the addition of the term (2.11) with $f(x)$ given correspondingly by (3.10) and (3.18) there is no duality transformation that relates them, because now both backgrounds depend explicitly on the angle ϕ and there is no isometry with respect to which dualization can be done. An exceptional case is for $\rho' = 0$, since then (3.10) does not depend on ϕ. Obviously, after dualization the new f is given by only the first term in (3.18) (computed at $\rho' = 0$), namely the ϕ-independent part, corresponding to the angular uniform distribution $\rho(r) = \frac{1}{2\pi r}\delta(r)$ (notice that, $\delta^{(2)}(x) = \frac{1}{r}\delta(r)\delta(\phi)$).

The solution for the shift function in the sourceless case and in a definite angular momentum sector is for $m = 0$ given: by $f \sim I_0(a\rho)$ if $\epsilon = 1$ and by $f \sim J_0(a\rho)$ if $\epsilon = -1$. For $m \neq 0$ the solution contains a confluent hypergeometric function, i.e., $f \sim \Phi\big(\frac{1}{2}(\frac{c}{2}+1), 1, \frac{1}{2}|m|\rho^2\big)\, e^{im\phi} e^{-\frac{1}{4}|m|\rho^2}$, which becomes (3.15) if c is one of the eigenvalues of the Laplacian.

3.3. $SL(2,\mathbb{R})_{-k}/\mathbb{R} \otimes SU(2)_{k'}/U(1)$

For this model the corresponding metric and dilaton are

$$\begin{aligned}
&ds^2 = \frac{2\epsilon}{uv-1}\, dudv + a^2\big(d\theta^2 + \tan^2\frac{\theta}{2}\, d\phi^2\big) \\
&\Phi = \phi_{||}(u,v) + \phi_{\perp}(\theta)\ , \qquad \phi_{||} = \ln(1-uv),\ \phi_{\perp} = \ln\cos^2\frac{\theta}{2} \\
&-\infty < u, v < \infty\ , \quad \phi \in [0,2\pi]\ , \quad \theta \in [0,\pi]\ ,
\end{aligned} \tag{3.21}$$

where $\epsilon = \mathrm{sign}(k)$ and $a^2 \equiv \frac{k'}{k}$ and similar physical interpretation as before. After we change variable as $x = \cos\theta$ equation (3.3) becomes

$$(1-x^2)\partial_x^2\Phi - 2x\partial_x\Psi + \frac{1+x}{1-x}\partial_\phi^2\Psi = -E\Psi\ . \tag{3.22}$$

Further substitution,

$$\Psi(x,\phi) = (1-x)^{|m|}e^{im\phi}T(x)\ , \quad m \in Z\ , \tag{3.23}$$

shows that $T(x)$ satisfies a Jacobi equation

$$(1-x^2)T'' - 2(|m| + (|m|+1)x)T' + (E-|m|)T = 0 \ . \tag{3.24}$$

A complete set of normalizable solutions to it exists (the so called Jacobi polynomials) provided that $E = n(n+1) + (2n+1)|m|$, where n is an integer. Using instead of n the non-negative integer $l = n + |m|$ we eventually get

$$\begin{aligned} \Psi_{l,m}(\theta,\phi) &= \sqrt{\frac{2l+1}{4\pi}}\ e^{im\phi}\ \sin^{2|m|}\frac{\theta}{2}\ P^{(2|m|,0)}_{l-|m|}(\cos\theta) \\ E_{l,m} &= l(l+1) - m^2\ , \quad l = 0,1,\dots,\ m = -l,-l+1,\dots,l\ , \end{aligned} \tag{3.25}$$

where a compatible with (3.4)(3.5) normalization factor has also been included. Notice that the eigenvalues $E_{l,m}$ are exactly what one expects from the coset construction for $SU(2)_k/U(1)$ for states at the base of the Virasoro modules and for high level k. The two terms that appear are the eigenvalues of the quadratic Casimirs for $SU(2)$ and $U(1)$ respectively. It is now straightforward to write down the solution for the shift function according to (3.2)

$$\begin{aligned} f(\theta,\phi;\theta',\phi') &= -b\sum_{l=0}^{\infty}\frac{l+\frac{1}{2}}{l(l+1)+c}P_l(\cos\theta)P_l(\cos\theta') \\ &- 2b\sum_{l=0}^{\infty}\sum_{m=1}^{l}\frac{(l+\frac{1}{2})\cos m(\phi-\phi')}{l(l+1)-m^2+c}\sin^{2m}\frac{\theta}{2}\sin^{2m}\frac{\theta'}{2}P^{(2m,0)}_{l-m}(\cos\theta)P^{(2m,0)}_{l-m}(\cos\theta')\ , \end{aligned} \tag{3.26}$$

where $c = \epsilon a^2$.

It should be possible to obtain expressions (3.10) (for $\alpha_0 = 0$) and (3.18) by taking appropriate limits in (3.26). The reason is that the corresponding spacetimes are related via limiting procedures. Specifically, if $\theta = \delta\rho$, $\phi \to 2\phi$ (that will effectively change $m \to m/2$ in (3.25)), $c \to c/\delta^2$ with $\delta \to 0$ the background (3.21) becomes that of (3.6) with $\alpha_0 = 0$. Naively the shift function as given by (3.26) becomes zero. However, if we treat carefully the contribution coming from the $l = k/\delta \to \infty$ values in the sum (which in this limit becomes an integral over k) and use

$$\lim_{l\to\infty}\left(\frac{x}{2l}\right)^{\alpha}P^{(\alpha,\beta)}_l(\cos\frac{x}{l}) = J_\alpha(x)\ , \tag{3.27}$$

for $l = \frac{k}{\delta}$, $x = \rho k$ and $\alpha = m$, $\beta = 0$ we obtain (3.9). Next we let $\theta = \pi - \delta\rho$, $\phi \to \phi\delta^2$, $c \to c/\delta^2$ with $\delta \to 0$. Because of the rescaling in ϕ the new ϕ will not be periodic and the corresponding eigenvalue $m \to m/\delta^2$ will not be an integer. To make ϕ again periodic we identify points in the real line, i.e., we quotient with a discrete subgroup of $\mathbb{R}$. Then the background (3.21) becomes that of (3.11). Then by letting $l = m/\delta^2 + n$ and using

$$
\begin{aligned}
&\lim_{\alpha\to\infty} P_n^{(\alpha,\beta)}(1 - \frac{2x}{\alpha}) = (-1)^n L_n^{\beta}(x) \\
&\lim_{\alpha\to\infty} \cos^{\alpha}(\sqrt{\frac{x}{\alpha}}) = e^{-x/2} \ ,
\end{aligned}
\tag{3.28}
$$

with $x = \frac{1}{2}m\rho^2$, $\alpha = 2m/\delta^2$, $\beta = 0$ we obtain from the double sum term of (3.26) a similar term in (3.18). Obviously the case $m = 0$ corresponding to the first term in (3.26) requires a different treatment since in this case we cannot take the $m \to \infty$ limit. In fact this term in (3.26) becomes $f_0(\rho;\rho')$ in (3.18) after using (3.27) for $l = \frac{k}{\delta}$, $x = \rho k$, $\alpha = \beta = 0$, replacing the sumation over l with an integral over the continuous variable k and evaluating this integral.

Let us finally mention that if the transverse CFT is the coset $SL(2,\mathbb{R})_{-k'}/\mathbb{R}$ corresponding to a 'Euclidian black hole' then we should analytically continue $\theta \to ir$ or $\theta \to \pi + ir$ and in (3.26) sum over the appropriate representations functions for the non-compact group $SL(2,\mathbb{R})$.

4. Concluding remarks and discussion

In this paper we investigated gravitational shock waves in string theory. We started with quite a general class of background solutions to the one loop β-function equations and found the conditions (see (2.6)(2.8)) that should be fulfilled in order to be able to introduce a shock wave via a coordinate shift. These shock waves may exist with or without sources. In the former case the source term was provided by tachyon fluctuations around a zero condensate value. In the sourceless case we rederived the same result by using CFT techniques and demanding that the relevant extra term in the $2d$ σ-model action (see

(2.11)) corresponds to a marginal perturbation (which was argued to be exactly marginal). In the case with sources the perturbation is not marginal by itself but it produces the necessary anomaly that cancels the term produced by the tachyon fluctuations, so that the combined model stays conformal. Moreover, the CFT method reveals that these conditions have the same form to all orders in α'. We also gave explicit results in some important 4-dimensional cases where the background geometry had an interpretation in terms of exact CFT's. Further utilization of the CFT method is done in appendix B (see (B.3)(B.4)).

From a string phenomenological point of view, the fact that random tachyon fluctuations give rise to gravitational shock waves is an important conclusion since the non-linear interactions of shock waves lead to interesting formations [28], including black holes (see, for instance, [6]). Questions of this nature should be further investigated.

It would also be interesting to consider scattering of particles and strings in the shock wave geometries we have obtained and in particular associate the results (for instance the pole structure of the S-matrix [9]) with the CFT properties of the corresponding backgrounds.

Acknowledgments

I would like to thank A.A. Tseytlin for bringing to my attention ref. [15].

Appendix A. Useful tensors

The non-vanishing Christoffel symbols corresponding to the metric (2.4) are

$$
\begin{aligned}
&\Gamma^u_{uu} = -\frac{F_{,v}}{2A} + \frac{A_{,u}}{A} \ , \qquad \Gamma^u_{ij} = -\frac{g_{,v}}{2A}\ h_{ij} \\
&\Gamma^v_{uu} = \frac{F_{,u}}{2A} + \frac{FF_{,v}}{2A^2} - \frac{FA_{,u}}{A^2} \ , \qquad \Gamma^v_{uv} = \frac{F_{,v}}{2A} \ , \qquad \Gamma^v_{ui} = \frac{F_{,i}}{2A} \\
&\Gamma^v_{vv} = \frac{A_{,v}}{A} \ , \qquad \Gamma^v_{ij} = \big(-\frac{g_{,u}}{2A} + \frac{Fg_{,v}}{2A^2}\big)\ h_{ij} \\
&\Gamma^i_{uu} = -\frac{1}{2g}\ h^{ik}F_{,k} \ , \qquad \Gamma^i_{uj} = \frac{g_{,u}}{2g}\ \delta^i_j \ , \qquad \Gamma^i_{vj} = \frac{g_{,v}}{2g}\ \delta^i_j \\
&\Gamma^i_{jk} = \frac{1}{2}h^{il}(h_{lk,j} + h_{lj,k} - h_{jk,l}) \ .
\end{aligned}
\tag{A.1}
$$

Using the above expressions we find that the non-vanishing components of the Ricci tensor are (we substitute $F = -2Af\delta$ (see (2.4)))

$$
\begin{aligned}
R_{uu} &= \frac{d-2}{2}\ \big(\frac{g_{,u}A_{,u}}{gA} - \frac{g_{,uu}}{g} + \frac{g^2_{,u}}{2g^2}\big) \ + \ \frac{A}{g}\ \delta\ \triangle_{h_{ij}} f \ - \ \frac{d-2}{2}\ \frac{g_{,v}}{g}\ \delta'\ f \\
&+ \ \big(2\ \frac{A_{,uv}}{A} - 2\ \frac{A_{,u}A_{,v}}{A^2} + \frac{d-2}{2gA}\ (g_{,u}A_{,v} + g_{,v}A_{,u})\big)\ \delta\ f \\
&+ \ 2\ \big(\frac{A_{,vv}}{A} - \frac{A^2_{,v}}{A^2} + \frac{d-2}{2}\ \frac{g_{,v}A_{,v}}{gA}\big)\ \delta^2\ f^2 \\
R_{uv} &= \big(\frac{A_{,u}A_{,v}}{A^2} - \frac{A_{,uv}}{A} + \frac{d-2}{4}\ \frac{g_{,u}g_{,v}}{g^2} - \frac{d-2}{2}\ \frac{g_{,uv}}{g}\big) \\
&+ \ \big(\frac{A^2_{,v}}{A^2} - \frac{A_{,vv}}{A} - \frac{d-2}{2}\ \frac{g_{,v}A_{,v}}{gA}\big)\ \delta\ f \\
R_{ui} &= -\big(\frac{d-4}{2}\ \frac{g_{,v}}{g} + \frac{A_{,v}}{A}\big)\ \delta\ f_{,i} \\
R_{vv} &= \frac{d-2}{2}\ \big(\frac{g_{,v}A_{,v}}{gA} + \frac{g^2_{,v}}{2g^2} - \frac{g_{,vv}}{g}\big) \\
R_{ij} &= R^{(d-2)}_{ij} \ - \ \big(\frac{d-4}{2}\ \frac{g_{,u}g_{,v}}{gA} + \frac{g_{,uv}}{A}\big)\ h_{ij} \ - \ \big(\frac{d-4}{2}\ \frac{g^2_{,v}}{gA} + \frac{g_{,vv}}{A}\big)\ h_{ij}\ \delta\ f \ .
\end{aligned}
\tag{A.2}
$$

The above expressions for the Christoffel symbols and the Ricci tensor were derived in [3]. For the reader's convenience we included them in this paper as well. The components of

$D_\mu D_\nu \Phi$ with $\Phi = \Phi(u,v,x)$ are given by

$$
\begin{aligned}
D_u D_u \Phi &= (\Phi_{,uu} - \frac{A_{,u}}{A}\Phi_{,u}) - \frac{1}{A}\ (A_{,u}\Phi_{,v} + A_{,v}\Phi_{,u})\ \delta\ f + \Phi_{,v}\ \delta'\ f \\
&- \ \frac{A}{g}\ \delta\ h^{ij} f_{,i}\Phi_{,j} - 2\frac{A_{,v}\Phi_{,v}}{A}\ \delta^2\ f^2 \\
D_u D_v \Phi &= \Phi_{,uv} + \frac{A_{,v}\Phi_{,v}}{A}\ \delta\ f \\
D_u D_i \Phi &= \Phi_{,ui} - \frac{g_{,u}}{2g}\ \Phi_{,i} + \Phi_{,v}\ f_{,i}\ \delta \\
D_v D_v \Phi &= \Phi_{,vv} - \frac{A_{,v}\Phi_{,v}}{A} \\
D_v D_i \Phi &= \Phi_{,vi} - \frac{g_{,v}}{2g}\ \Phi_{,i} \\
D_i D_j \Phi &= \Phi_{,ij} - \Gamma^k_{ij}\Phi_{,k} + \frac{1}{2A}\ (g_{,u}\Phi_{,v} + g_{,v}\Phi_{,u})\ h_{ij} + \frac{g_{,v}\Phi_{,v}}{A}\ h_{ij}\ \delta\ f\ .
\end{aligned}
\tag{A.3}
$$

We also compute

$$
\begin{aligned}
(H^2)_{uu} &= \frac{1}{g^2}H_{uij}H_{ukl}h^{ki}h^{lj} + \frac{4}{gA}H_{uvi}H_{uvj}h^{ij}f\delta \\
(H^2)_{uv} &= \frac{1}{g^2}H_{uij}H_{vkl}h^{ki}h^{lj} - \frac{2}{gA}H_{uvi}H_{uvj}h^{ij} \\
(H^2)_{vv} &= \frac{1}{g^2}H_{vij}H_{vkl}h^{ki}h^{lj} \\
(H^2)_{ui} &= \frac{1}{g^2}H_{ujl}H_{imn}h^{jm}h^{ln} - \frac{2}{gA}H_{uvj}H_{uim}h^{mj} - \frac{4}{Ag}H_{uvj}H_{uim}h^{mj}f\delta \\
(H^2)_{vi} &= \frac{1}{g^2}H_{vjl}H_{imn}h^{mj}h^{nl} + \frac{2}{gA}H_{uvj}H_{vim}h^{mj} \\
(H^2)_{ij} &= \frac{1}{g^2}H_{ikl}H_{jmn}h^{km}h^{ln} - \frac{2}{A^2}H_{iuv}H_{juv} \\
&\quad + \frac{2}{gA}\big(H_{imu}H_{jnv} + H_{imv}H_{jnu} + 2H_{imv}H_{jnv}f\delta\big)h^{mn}\ .
\end{aligned}
\tag{A.4}
$$

Appendix B. Shock waves on more general string backgrounds

In this appendix we construct shock waves on more general than (2.1) string backgrounds using the new method based on CFT techniques that was introduced in section 2. Consider the string background

$$
\begin{aligned}
ds^2 &= 2\ A(u,v,x)\ dudv + E(u,v,x)\ du^2 + g_{ij}(u,v,x)\ dx^i dx^j \\
B &= 2B_{ui}(u,v,x)\ du \wedge dx^i + B_{ij}(u,v,x)\ dx^i \wedge dx^j \\
\Phi &= \Phi(u,v,x)\ .
\end{aligned}
\tag{B.1}
$$

Let us add to the σ-model action corresponding to that a similar to (2.11) term

$$-2\int d^2z\ \delta(u)A(u,v,x)\lambda(u,v,x)\ \partial u\bar{\partial}u\ . \tag{B.2}$$

Notice that because of the dependence of $\lambda(u,v,x)$ on the extra coordinates u,v this term cannot be obtained, in general, via a simple shift of v as (2.11). Demanding that $\partial u\bar{\partial}u$ has dimension $(1,1)$ with respect to the energy momentum tensor corresponding to (B.1) imposes the conditions

$$A_{,v}=E_{,v}=g_{ij,v}=\Phi_{,v}=\lambda_{,v}=0\quad \text{at } u=0\ . \tag{B.3}$$

Notice that, among other differences with (2.6), there is now a condition on the function $\lambda(u,v,x)$ itself. Demanding that $\lambda(u,v,x)\delta(u)$ transforms like a function and adding the source term (due, for instance, to stochastic tachyon fluctuations) we obtain the linear differential equation

$$\begin{aligned}&\triangle\lambda(0,v,x)-c(v,x)\lambda(0,v,x)=2\pi k\ \delta^{(d-2)}(x-x')\\&c(v,x)\equiv\frac{1}{A}(\frac{1}{2}g^{ij}g_{ij,uv}+\Phi_{,uv})\ ,\end{aligned} \tag{B.4}$$

where all functions are computed at $u=0$ and the Laplacian is defined as (cf.(2.9))

$$\triangle=\frac{1}{e^{\Phi}\sqrt{g}A}\partial_i e^{\Phi}\sqrt{g}A\ g^{ij}\partial_j\ . \tag{B.5}$$

It is easy to see that (B.3)(B.5) reduce to (2.6)(2.8) when the background (B.1) specializes to (2.1). In the cases where $E(u,v,x)$ is zero at $u=0$, arguments similar to (2.21)(2.22) show that the term (B.2) corresponds to an exactly marginal perturbation.

An example of a background belonging to the more general class (B.1) is given by

$$\begin{aligned}&ds^2=2V^{-1}(dudv+dzd\bar{z})\ ,\quad V^{-1}=2C+(uv+z\bar{z})^{-1}\\&\Phi=\ln V\ ,\quad H_{\mu\nu\rho}=2\epsilon_{\mu\nu\rho\lambda}\partial^{\lambda}\Phi\ ,\end{aligned} \tag{B.6}$$

where C is a constant. When $C=0$ this corresponds to the direct product CFT of $SU(2)_k$ with a timelike boson having a background charge [29] or the Minkowski continuation of

the 'semi-wormhole' model, with $N = 4$ worldsheet supersymmetry, of [30]. It is easy to show that (B.4) reduces to $\triangle\lambda = 2\pi k\delta^{(2)}(z - z')$, i.e., the same as in the flat space case, with solution $\lambda = k \ln |z-z'|$. This is not surprising since the Einstein metric corresponding to (B.6) is the Minkowski one.

It is important (in order to exclude any surprises) to verify that (B.3)(B.4) also follow by requiring that the β-function equations are satisfied for the generic background (B.1). Such a tedious computation has been explicitly performed and the result is exactly (B.3)(B.4).

Appendix C. Elements of stochastic calculus

In this appendix we present some elementary facts of stochastic calculus that are needed in order to prove (2.18)-(2.20). For more details the reader should consult one of the many relevant books and review articles in the literature (see, for instance, [31]).

A stochastic integral of a representative function $h(t)$ of a class of stochastic processes is defined as

$$I(t;t_0) = \int_{t_0}^{t} dw(t)\ h(t) \equiv \lim_{n\to\infty} \sum_{k=1}^{n} h(\tau_k)\big(w(t_k) - w(t_{k-1})\big)\ , \tag{C.1}$$

where τ_k is a point in the interval $[t_{k-1}, t_k]$. The stochastic variable $w(t)$ associated with a Brownian motion satisfies the properties

$$\langle w(t)\rangle = w_0\ , \qquad \langle w(t)w(s)\rangle = (s - t_0 + w_0^2)\ \vartheta(t-s) + (t - t_0 + w_0^2)\ \vartheta(s-t)\ , \tag{C.2}$$

from which one easily proves that the stochastic differential $dw(t)$ obeys

$$\langle dw(t)\rangle = 0\ , \qquad \langle dw(t)dw(s)\rangle = \delta(t-s)dtds\ . \tag{C.3}$$

The second relation states that $(dw(t))^2 = O(dt)$ and therefore $dw(t)$ should be treated as a differential of order $\frac{1}{2}$ in various algebraic manipulations and Taylor expansions. It turns out that the integral $I(t;t_0)$ is not independent of the choice of the intermediate

point $\tau_k \in [t_{k-1}, t_k]$. The choice $\tau_k = t_{k-1}$ corresponds to the *Ito calculus*. For the average of two Ito integrals corresponding to two stochastic functions $h_1(t)$, $h_2(t)$ the formula

$$\langle \int_{t_0}^{t} dw(t)\ h_1(t) \int_{t_0}^{t} dw(s)\ h_2(s) \rangle = \int_{t_0}^{t} dt\ \langle h_1(t) h_2(t) \rangle\ , \tag{C.4}$$

is extremely useful because it converts a double integration over the stochastic variable $w(t)$ into a single ordinary integral. For the proof of (C.4) the definition (C.1) and (C.2) should be used together with the crucial assumption that the stochastic functions $h_1(t)$, $h_2(t)$ are independent of $w(s)$ for $t < s$, namely, that

$$\langle h_i(t) w(s) \rangle = 0, \quad i = 1, 2\ , \quad \text{if } t < s\ . \tag{C.5}$$

Obviously, if $h_1(t)$, $h_2(t)$ are deterministic functions there is no need to take the average on the right hand side of (C.4). This is the case in the derivation of (2.20) in section 2.

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